

**Higher-Order Gravitational Perturbations
of the Cosmic Microwave Background ***

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Abstract

We study the behavior of light rays in perturbed Robertson-Walker cosmologies, calculating the redshift between an observer and the surface of last scattering to second order in the metric perturbation. At first order we recover the classic results of Sachs and Wolfe, and at second order we delineate the various new effects which appear; there is no *a priori* guarantee that these effects are significantly smaller than those at first order, since there are large length scales in the problem which could lead to sizable prefactors. We find that second order terms of potential observational interest may be interpreted as transverse and longitudinal lensing by foreground density perturbations, and a correction to the integrated Sachs-Wolfe effect.

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I. Introduction

In the last several years, observations of temperature anisotropies in the cosmic microwave background (CMB) [1] have spurred increasingly sophisticated investigation of the anisotropy predicted by theoretical models [2-5]. Important contributions to the anisotropy come from gravitational perturbations, temperature and pressure fluctuations at the surface of last scattering, and ionization effects in the later universe.

The earliest of these effects to be studied, and the most important on large scales, are those due to gravitational perturbations. These were systematically investigated by Sachs and Wolfe [6], who derived the basic formulae relating perturbations in the metric to anisotropy in the temperature of the CMB. Their results revealed two basic sources of anisotropy: potential fluctuations at the surface of last scattering, and time variation of the potential along the path of the photon. Later investigations focused on individual effects in specific models [7-15].

Even though perturbations in the energy density $\delta\rho/\rho$ grow to be greater than unity on sufficiently small scales, the resulting metric perturbations may almost always be taken to be small [16]. It therefore makes sense to calculate the behavior of photons to first order in this perturbation, as Sachs and Wolfe did. Nevertheless, there is no way of knowing ahead of time that second-order terms in an expansion in the metric perturbation will be negligible compared to the first order terms, since there is ample opportunity for effects to accumulate as photons travel to the observer from the surface of last scattering; in other words, the coefficients of the second-order terms may be numerically large. (As an example of a related effect, the time delay formula in standard gravitational lens systems contains important contributions from both the first order Shapiro and the second order geometric effects.) It is therefore worthwhile investigating the redshift induced by effects which are formally second order in the metric perturbation to see if they may nevertheless be observationally important. In this paper we calculate these second-order effects and interpret the results in terms of specific physical processes.

It is necessary to be careful about what we mean by “second order” in the context of gravitational perturbation theory. We imagine that we are given a metric throughout spacetime of the form

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu} \ , \tag{1.1}$$

where $g_{\mu\nu}^{(0)}$ describes a background Robertson-Walker spacetime and $h_{\mu\nu}$ is a small per-

turbation. We will not be computing this perturbation to second order in perturbations of the energy-momentum tensor, but simply calculating photon trajectories to second order in $h_{\mu\nu}$ and its derivatives. Therefore, if $h_{\mu\nu}$ is computed from standard first-order metric perturbation theory and substituted into our expressions, the results will not represent a complete calculation of effects which are second order in the matter perturbations. (In Sec. IV we will examine explicitly the case of first-order scalar perturbations, but it is straightforward to generalize the results.) Nevertheless the expressions we obtain will constitute a subset of all the possible contributions, and if any of them turn out to be comparable in magnitude to terms which are formally first order, it is appropriate to take them into account. Moreover, the substitution $h_{\mu\nu} \mapsto g_{\mu\nu}^{(1)} + g_{\mu\nu}^{(2)} + \dots$ into our formulae below would immediately yield an expansion for the full second order anisotropy.

It is also important to note that we will only be dealing with gravitational perturbations. We will imagine that there is a hypersurface of last scattering fixed at some definite time, on which there can exist intrinsic perturbations which may be calculated independently; we then compute the additional perturbations due to the metric fluctuations along the geodesics followed by the photons. Non-gravitational second-order perturbations were treated by Vishniac [17], Dodelson and Jubas [18], and Hu, Scott and Silk [19]. The latter authors also examined higher order gravitational effects by expanding the Boltzmann equation to second order, but did not construct explicit solutions. As a final caveat, we treat the order-by-order expansion in powers of the metric perturbation and its derivatives in a formal sense; thus, a phenomenon such as the integrated Sachs-Wolfe (or Rees-Sciama) effect we consider to be first order (since it involves terms linear in derivatives of $h_{\mu\nu}$), even though it is sometimes thought of as second order since it can be numerically small (and vanishes to first order in some specific models).

Our calculation proceeds as follows. In Sec. II we set up the problem and express the redshift experienced by a photon in terms of its corresponding background path $x^{(0)\mu}(\lambda)$ and its first and second order perturbations, $x^{(1)\mu}(\lambda)$ and $x^{(2)\mu}(\lambda)$. In Sec. III we discuss a general formalism for constructing these perturbations in terms of the metric variables; this is an extension of the methods of Pyne and Birkinshaw [20] to arbitrary order. In Sec. IV we specialize to the case of scalar perturbations, and examine the resulting formula for the temperature anisotropy. Although a quantitative understanding of the magnitude of each term would require detailed knowledge of the evolution of the metric perturbations (which we do not attempt in this paper), it is possible to discuss informally which contributions

might be observable in realistic models of structure formation.

II. Perturbation Expansion

We are interested in the pattern of temperature fluctuations $\Delta T/T$ on the sky as seen by an observer in a perturbed Robertson-Walker spacetime. We write our background metric in conformal coordinates $x^\mu = (\eta, x, y, z)$ as

$$\begin{aligned} d\bar{s}^{(0)} &= \bar{g}_{\mu\nu}^{(0)} dx^\mu dx^\nu \\ &= a^2(\eta)[-d\eta^2 + \gamma^{-2}(dx^2 + dy^2 + dz^2)] . \end{aligned} \quad (2.1)$$

Here $\gamma = 1 + \kappa r^2/4$, where κ is the spatial curvature parameter (+1, -1 and 0 for positively curved, negatively curved and flat cases, respectively), $a(\eta)$ is the scale factor, and $r^2 = x^2 + y^2 + z^2$. In this section we consider an arbitrary metric perturbation $\bar{h}_{\mu\nu}$. It will be convenient to separate out the dependence on the scale factor by working in the conformal background metric $g_{\mu\nu}^{(0)} = a^{-2}\bar{g}_{\mu\nu}^{(0)}$, with the conformally-transformed perturbation $h_{\mu\nu} = a^{-2}\bar{h}_{\mu\nu}$, so that the actual, physical spacetime metric is given by $\bar{g}_{\mu\nu} = \bar{g}_{\mu\nu}^{(0)} + \bar{h}_{\mu\nu}$. The wavevector \bar{k}^μ of a light ray in the physical metric is related to the wavevector k^μ in the conformally transformed metric by $k^\mu = a^2\bar{k}^\mu$. (Our conventions are those of Ref. [20].)

Within such a spacetime we consider a photon path $x^\mu(\lambda)$, where λ is an affine parameter. (See Fig. 1.) This path connects an observer at a point \mathcal{O} with coordinates $x_{\mathcal{O}}^\mu = (\eta_{\mathcal{O}}, 0, 0, 0)$ to the hypersurface of emission, which we define to be the spacelike hypersurface of constant conformal time $\eta = \eta_{\mathcal{E}}$. The “surface” of emission is then the intersection of the past light cone of the observer with this hypersurface. We assume that at conformal time $\eta_{\mathcal{E}}$ every point with spatial coordinates p^i emits thermal radiation with a temperature $T_{\mathcal{E}}(\mathbf{p}, \hat{\mathbf{d}})$, as measured by a comoving observer, which depends both on position and on direction as characterized by a three-vector $\hat{\mathbf{d}}$, normalized to unity in the background metric, $g_{\mu\nu}^{(0)}$, restricted to the hypersurface. (This hypersurface need not be the actual time of last scattering, but need only represent a hypersurface on which the radiation field is understood.) The photon path itself is specified by a three-vector $\hat{\mathbf{e}}$ in the hypersurface of constant conformal time containing \mathcal{O} normalized to unity in $g_{\mu\nu}^{(0)}$. We can think of $\hat{\mathbf{e}}$ as the direction on the sky toward which a comoving observer at \mathcal{O} is pointing an antenna; for observers which are not comoving $\hat{\mathbf{e}}$ and the observer’s direction vector are related by a Lorentz transformation. The initial condition $\hat{\mathbf{e}}$ determines the point \mathbf{p} and direction vector $\hat{\mathbf{d}}$ at which the ray intersects the hypersurface of emission.

To an observer with four-velocity U^μ (normalized to $U^\mu \bar{g}_{\mu\nu} U^\nu = -1$), a photon with wavevector $k^\nu = dx^\nu/d\lambda$, with λ an affine parameter in the conformal metric $g_{\mu\nu}$, has a relative frequency given by

$$\omega = -a^{-2} \bar{g}_{\mu\nu} U^\mu k^\nu . \quad (2.2)$$

(We refer to this as the “relative” frequency, since we are free to scale the affine parameter λ to set the normalization of ω . The ratio of relative frequencies at two points along the path is invariant under such a reparameterization.) For a blackbody spectrum, the CMB temperature observed at \mathcal{O} is related to the temperature at emission by

$$T_{\mathcal{O}}(\mathbf{x}_{\mathcal{O}}, \hat{\mathbf{e}}) = \frac{\omega_{\mathcal{O}}}{\omega_{\mathcal{E}}} T_{\mathcal{E}}(\mathbf{p}, \hat{\mathbf{d}}) . \quad (2.3)$$

We are therefore interested in computing, given the initial data $x_{\mathcal{O}}^\mu, \hat{\mathbf{e}}, \omega_{\mathcal{O}}$, the quantities $\mathbf{p}, \hat{\mathbf{d}}$, and $\omega_{\mathcal{E}}$. These depend on the photon path and associated wavevector, which we may express as series expansions in the perturbation $h_{\mu\nu}$ and its derivatives:

$$\begin{aligned} x^\mu(\lambda) &= x^{(0)\mu}(\lambda) + x^{(1)\mu}(\lambda) + x^{(2)\mu}(\lambda) + \dots \\ k^\mu(\lambda) &= k^{(0)\mu}(\lambda) + k^{(1)\mu}(\lambda) + k^{(2)\mu}(\lambda) + \dots \end{aligned} \quad (2.4)$$

The situation is thus as portrayed in Fig. 1. Note that $x^{(0)\mu}(\lambda)$ has the interpretation of a path through spacetime, while the $x^{(a)\mu}(\lambda)$ are thought of as deviation vectors at each λ . In this section we will calculate the observed temperature in terms of these quantities (plus the intrinsic temperature fluctuations on the surface of emission), while in the next section we will explicitly calculate the path and wavevector in terms of the metric perturbation.

We have already specified $T_{\mathcal{E}}$ as the temperature measured by a comoving observer. It will also be convenient to take our observer at \mathcal{O} comoving. This requirement is physically acceptable, since any motion of the observer leads to a dipole anisotropy which may be easily subtracted. It is sometimes useful to imagine a family of comoving observers with four-velocity U^μ defined over all of spacetime. The normalization condition $U^\mu \bar{g}_{\mu\nu} U^\nu = -1$ then leads to

$$\begin{aligned} U^{(0)\mu} &= a^{-1}(1, 0, 0, 0) \\ U^{(1)\mu} &= a^{-1} \left(\frac{1}{2} h_{00}, 0, 0, 0 \right) \\ U^{(2)\mu} &= a^{-1} \left(\frac{3}{8} (h_{00})^2, 0, 0, 0 \right) . \end{aligned} \quad (2.5)$$

We can also explicitly construct the geodesics of the background metric, $x^{(0)\mu}(\lambda)$. We consider null rays which intersect the observer at the spatial origin of co-ordinates, and we choose the affine parameter such that

$$\begin{aligned} k^{(0)0} &= 1 , \\ g_{ij}^{(0)} k^{(0)i} k^{(0)j} &= 1 . \end{aligned} \tag{2.6}$$

A two parameter family of such rays which satisfy these conditions is given by [21]

$$\begin{aligned} x^{(0)\mu} &= (\lambda, r e^i) \\ k^{(0)\mu} &= (1, -\gamma e^i) , \end{aligned} \tag{2.7}$$

where the e^i are components of $\hat{\mathbf{e}}$, and

$$\begin{aligned} r(\lambda) &= 2 \tan_{\kappa} \left(\frac{\lambda_{\mathcal{O}} - \lambda}{2} \right) \\ \gamma(\lambda) &= \sec_{\kappa}^2 \left(\frac{\lambda_{\mathcal{O}} - \lambda}{2} \right) , \end{aligned} \tag{2.8}$$

where $\lambda_{\mathcal{O}}$ is the affine parameter at the observer. The subscript κ on a trigonometric function denotes a set of three functions: for $\kappa = 1$ the trigonometric function itself, for $\kappa = -1$ the corresponding hyperbolic function, and for $\kappa = 0$ the first term in the series expansion of the function. (Thus, $\sin_0 \theta = \theta$, $\cos_0 \theta = 1$.) Finally, we can place boundary conditions on the higher-order quantities $x^{(1)\mu}$, $x^{(2)\mu}$, $k^{(1)\mu}$ and $k^{(2)\mu}$ at the origin. For convenience we will set

$$\begin{aligned} x^{(1)\mu}(\lambda_{\mathcal{O}}) &= x^{(2)\mu}(\lambda_{\mathcal{O}}) = 0 \\ k^{(1)i}(\lambda_{\mathcal{O}}) &= k^{(2)i}(\lambda_{\mathcal{O}}) = 0 . \end{aligned} \tag{2.9}$$

Then the condition that the wavevector be null at the observer implies that

$$\begin{aligned} k^{(1)0}(\lambda_{\mathcal{O}}) &= \left(\frac{1}{2} h_{00} + h_{0i} k^{(0)i} + \frac{1}{2} h_{ij} k^{(0)i} k^{(0)j} \right)_{\mathcal{O}} , \\ k^{(2)0}(\lambda_{\mathcal{O}}) &= \left[\frac{3}{8} (h_{00})^2 + h_{00} h_{0i} k^{(0)i} + \frac{1}{4} h_{00} h_{ij} k^{(0)i} k^{(0)j} + \frac{1}{2} (h_{0i} k^{(0)i})^2 \right. \\ &\quad \left. - \frac{1}{8} (h_{ij} k^{(0)i} k^{(0)j})^2 \right]_{\mathcal{O}} . \end{aligned} \tag{2.10}$$

The temperature at emission can be written as a uniform background plus a small perturbation, expressed as

$$T_{\mathcal{E}}(\mathbf{p}, \hat{\mathbf{d}}) = [1 + \tau(\mathbf{p}, \hat{\mathbf{d}})] T_{\mathcal{E}}^{(0)} . \tag{2.11}$$

The function τ will be treated as first order (*i.e.* of the same order as $h_{\mu\nu}$), and will be unspecified in this paper since our interest is in the gravitational effects on photons in the time since emission. The point at which the geodesic intersects the surface $\eta = \eta_{\mathcal{E}}$ can be written as $\mathbf{p} = \mathbf{p}^{(0)} + \mathbf{p}^{(1)} + \dots$ (Note the distinction between x^i , the spacelike components of the separation vector, and p^i , the separation of the intersection points of the path at different orders with the constant-time hypersurface.) Expanding $\omega_{\mathcal{E}}$ and $\hat{\mathbf{d}}$ as well, eq. (2.3) to second order becomes

$$T_{\mathcal{O}} = \frac{\omega_{\mathcal{O}}^{(0)} + \omega_{\mathcal{O}}^{(1)} + \omega_{\mathcal{O}}^{(2)}}{\omega_{\mathcal{E}}^{(0)} + \omega_{\mathcal{E}}^{(1)} + \omega_{\mathcal{E}}^{(2)}} [1 + \tau(\mathbf{p}^{(0)} + \mathbf{p}^{(1)}, \hat{\mathbf{d}}^{(0)} + \hat{\mathbf{d}}^{(1)})] T_{\mathcal{E}}^{(0)}. \quad (2.12)$$

With the conventions chosen in the previous paragraph, $\omega_{\mathcal{O}}^{(0)} = a(\eta_{\mathcal{O}})^{-1}$ and $\omega_{\mathcal{E}}^{(0)} = a(\eta_{\mathcal{E}})^{-1}$. The quantity of interest to us is the fractional deviation in the observed temperature with respect to the expected temperature in the unperturbed spacetime, and we denote this deviation by $\widehat{\delta T}$. Expanding τ in a Taylor series, we obtain

$$\begin{aligned} \widehat{\delta T} &\equiv \left(\frac{\omega_{\mathcal{E}}^{(0)}}{\omega_{\mathcal{O}}^{(0)}} \right) \frac{T_{\mathcal{O}}}{T_{\mathcal{E}}^{(0)}} \\ &= \left[1 + \left(\tilde{\omega}_{\mathcal{O}}^{(1)} - \tilde{\omega}_{\mathcal{E}}^{(1)} + \tau \right) \right. \\ &\quad \left. + \left(\tilde{\omega}_{\mathcal{O}}^{(2)} - \tilde{\omega}_{\mathcal{E}}^{(2)} + (\tilde{\omega}_{\mathcal{E}}^{(1)})^2 - \tilde{\omega}_{\mathcal{O}}^{(1)} \tilde{\omega}_{\mathcal{E}}^{(1)} + \tilde{\omega}_{\mathcal{O}}^{(1)} \tau - \tilde{\omega}_{\mathcal{E}}^{(1)} \tau + p^{(1)i} \frac{\partial \tau}{\partial x^i} + d^{(1)i} \frac{\partial \tau}{\partial d^i} \right) \right], \end{aligned} \quad (2.13)$$

where the d^i are the components of $\hat{\mathbf{d}}$, τ and its first partial derivatives are evaluated at $(\mathbf{p}^{(0)}, \hat{\mathbf{d}}^{(0)})$, and we have put $\tilde{\omega}^{(a)} = \omega^{(a)}/\omega^{(0)}$. We note that our freedom to choose $T_{\mathcal{E}}^{(0)}$ may be used to render $\widehat{\delta T}$ observable, *e.g.* by setting $T_{\mathcal{E}}^{(0)} = a(\eta_{\mathcal{O}}) a(\eta_{\mathcal{E}})^{-1} \langle T_{\mathcal{O}} \rangle$ where the angle brackets denote an average over the observer's sky.

Expanding the metric perturbation and photon wavevector around their values on the background path, we obtain

$$\begin{aligned} \tilde{\omega}^{(0)} &= 1 \\ \tilde{\omega}^{(1)} &= -\frac{1}{2} h_{00} - k^{(0)i} h_{0i} + k^{(1)0} \\ \tilde{\omega}^{(2)} &= -\frac{1}{8} (h_{00})^2 - \frac{1}{2} h_{00} k^{(1)0} - \frac{1}{2} k^{(0)i} h_{0i} h_{00} - h_{0i} k^{(1)i} + k^{(2)0} \\ &\quad - \frac{1}{2} p^{(1)i} \frac{\partial h_{00}}{\partial x^i} - k^{(0)i} p^{(1)j} \frac{\partial h_{0i}}{\partial x^j} + \Delta \lambda \frac{dk^{(1)0}}{d\lambda} - h_{0i} \Delta \lambda \frac{dk^{(0)i}}{d\lambda}. \end{aligned} \quad (2.14)$$

In this expression $\Delta\lambda$ is the difference in affine parameter between the point where the zeroth and first order geodesics intersect the hypersurface $\eta = \text{constant}$; to this order $\Delta\lambda = -x^{(1)0}$. It is also straightforward to show that $p^{(1)i} = x^{(1)i} - k^{(0)i}x^{(1)0}$, and that $d^{(1)i}$ is given by

$$d^{(1)i} = \frac{k^{(0)i} + k^{(1)i}}{|k^{(0)i} + k^{(1)i}|} - \frac{k^{(0)i}}{|k^{(0)i}|}, \quad (2.15)$$

where the norm is defined by the spacelike part of the background metric. Putting it all together we obtain

$$\begin{aligned} \widehat{\delta T}^{(0)} &= 1 \\ \widehat{\delta T}^{(1)} &= \left[\frac{1}{2} h_{ij} k^{(0)i} k^{(0)j} \right]_{\mathcal{O}} + \left[\frac{1}{2} h_{00} + h_{0i} k^{(0)i} - k^{(1)0} + \tau \right]_{\bar{\mathcal{E}}} \\ \widehat{\delta T}^{(2)} &= \left[\frac{1}{2} (h_{0i} k^{(0)i})^2 - \frac{1}{8} (h_{ij} k^{(0)i} k^{(0)j})^2 \right]_{\mathcal{O}} \\ &\quad + \left[\frac{1}{2} h_{ij} k^{(0)i} k^{(0)j} \right]_{\mathcal{O}} \left[\frac{1}{2} h_{00} + h_{0i} k^{(0)i} - k^{(1)0} + \tau \right]_{\bar{\mathcal{E}}} \\ &\quad + \left[\frac{3}{8} (h_{00})^2 - \frac{1}{2} h_{00} k^{(1)0} + \frac{3}{2} h_{0i} h_{00} k^{(0)i} + (h_{0i} k^{(0)i})^2 - 2 h_{0i} k^{(0)i} k^{(1)0} \right. \\ &\quad \left. + h_{0i} k^{(1)i} + (k^{(1)0})^2 - k^{(2)0} + \left(\frac{1}{2} h_{00} + h_{0i} k^{(0)i} - k^{(1)0} \right) \tau + x^{(1)0} \frac{dk^{(1)0}}{d\lambda} \right. \\ &\quad \left. - h_{0i} x^{(1)0} \frac{dk^{(0)i}}{d\lambda} + (x^{(1)i} - k^{(0)i} x^{(1)0}) \left(\frac{1}{2} \frac{\partial h_{00}}{\partial x^i} + k^{(0)j} \frac{\partial h_{0j}}{\partial x^i} + \frac{\partial \tau}{\partial x^i} \right) + d^{(1)i} \frac{\partial \tau}{\partial d^i} \right]_{\bar{\mathcal{E}}}. \end{aligned} \quad (2.16)$$

Here, the notation $\bar{\mathcal{E}}$ means that the quantities referred to should be evaluated at the point $(\eta_{\mathcal{E}}, \mathbf{p}^{(0)})$ and direction $\hat{\mathbf{d}}^{(0)}$.

To complete the above formulae, we have to solve for the perturbed geodesics at first and second order in terms of $h_{\mu\nu}$. In the next section we carry this out for arbitrary metric perturbations, and in the following section we specialize to scalar perturbations.

III. Second-Order Geodesics

In order to calculate the approximate geodesics of $g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}$ order by order we employ the perturbative geodesic expansion introduced in Pyne and Birkinshaw [20]. Because those authors worked only to first order it is necessary slightly to extend the equations to address the higher order questions we are concerned with here. In this section

we describe the needed extension, which writes a general solution for the approximate path at any order without restriction on the perturbed spacetime under consideration. In the following section we specialize this general solution to gain the null geodesics to second order of perturbed FRW spacetimes in the longitudinal gauge.

We begin with the geodesic equation in the metric $g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}$,

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} k^\alpha k^\beta = 0, \quad (3.1)$$

which holds along some path $x^\mu(\lambda)$. We seek to approximate that path to any given order by solving for the $x^{(a)\mu}(\lambda)$ in (2.4). To this end we substitute (2.4) and the equation

$$\Gamma^\mu_{\alpha\beta} = \Gamma^{(0)\mu}_{\alpha\beta} + \Gamma^{(1)\mu}_{\alpha\beta} + \Gamma^{(2)\mu}_{\alpha\beta} + \dots \quad (3.2)$$

into (3.1) and simultaneously Taylor expand each of the $\Gamma^{(a)\mu}_{\alpha\beta}$ at $x^\mu(\lambda)$ about their value at $x^{(0)\mu}(\lambda)$. In (3.2), $\Gamma^{(a)\mu}_{\alpha\beta}$ is that part of $\Gamma^\mu_{\alpha\beta}$ which is of a -th order in either $h_{\mu\nu}$, its first partial derivatives, or their products. The resulting equation, equivalent to (3.1) but holding along the path $x^{(0)\mu}(\lambda)$, is written

$$\begin{aligned} \Sigma_{a=0}^\infty \left[\frac{d^2 x^{(a)\mu}}{d\lambda^2} + \left(\Gamma^{(a)\mu}_{\alpha\beta} + \Sigma_{b=1}^\infty \frac{1}{b!} \partial_{\sigma_1} \cdots \partial_{\sigma_b} \Gamma^{(a)\mu}_{\alpha\beta} \left(\Sigma_{c=1}^\infty x^{(c)\sigma_1} \right) \cdots \left(\Sigma_{d=1}^\infty x^{(d)\sigma_l} \right) \right) \right. \\ \left. \times \left(\Sigma_{e=1}^\infty k^{(e)\alpha} \right) \left(\Sigma_{f=1}^\infty k^{(f)\beta} \right) \right] = 0 \end{aligned} \quad (3.3)$$

At zeroth order we find that $x^{(0)\mu}(\lambda)$ is an affinely parametrized geodesic in the metric $g_{\mu\nu}^{(0)}$. At every order above zeroth equation (3.3) may be rearranged into the form of a forced Jacobi equation for the a -th order separation vector, $x^{(a)\mu}(\lambda)$:

$$\frac{d^2 x^{(a)\mu}}{d\lambda^2} + 2\Gamma^{(0)\mu}_{\alpha\beta} k^{(0)\alpha} k^{(a)\beta} + \partial_\sigma \Gamma^{(0)\mu}_{\alpha\beta} k^{(0)\alpha} k^{(0)\beta} x^{(a)\sigma} = f^{(a)\mu}. \quad (3.4)$$

Importantly, the highest order $x^{(b)\mu}$ or $k^{(b)\mu}$ appearing in $f^{(a)\mu}$ is of $(a-1)$ -th order. For instance, the forcing vectors at first and second order are given by

$$\begin{aligned} f^{(1)\mu} &= -\Gamma^{(1)\mu}_{\alpha\beta} k^{(0)\alpha} k^{(0)\beta} \\ f^{(2)\mu} &= -\Gamma^{(0)\mu}_{\alpha\beta} k^{(1)\alpha} k^{(1)\beta} - 2\Gamma^{(1)\mu}_{\alpha\beta} k^{(0)\alpha} k^{(1)\beta} - 2\partial_\sigma \Gamma^{(0)\mu}_{\alpha\beta} x^{(1)\sigma} k^{(0)\alpha} k^{(1)\beta} \\ &\quad - \partial_\sigma \Gamma^{(1)\mu}_{\alpha\beta} x^{(1)\sigma} k^{(0)\alpha} k^{(0)\beta} - \frac{1}{2} \partial_\sigma \partial_\tau \Gamma^{(0)\mu}_{\alpha\beta} x^{(1)\sigma} x^{(1)\tau} k^{(0)\alpha} k^{(0)\beta} \\ &\quad - \Gamma^{(2)\mu}_{\alpha\beta} k^{(0)\alpha} k^{(0)\beta}. \end{aligned} \quad (3.5)$$

Pyne and Birkinshaw [20] showed how a general solution to equation (3.4) could be written down in terms of the parallel and Jacobi propagators of the background metric $g_{\mu\nu}^{(0)}$. These propagators are matrix-valued functions of a pair of points connected by a geodesic, and are defined by path-ordered exponentials along the appropriate geodesic. The parallel propagator $P(\lambda_1, \lambda_2)^\mu{}_\nu$ is a 4×4 matrix given by

$$P(\lambda_2, \lambda_1) = \mathcal{P} \exp \left(-\frac{1}{2} \int_{\lambda_1}^{\lambda_2} A(\lambda) d\lambda \right), \quad (3.6)$$

where \mathcal{P} denotes the path ordering symbol and A is a 4×4 matrix defined by $A^\mu{}_\nu = 2k^{(0)\sigma}\Gamma^{(0)\mu}{}_{\sigma\nu}$. The parallel propagator lives up to its name, in that $P(\lambda_1, \lambda_2)^\mu{}_\nu v^\nu(\lambda_2)$ is the vector obtained by parallel propagating v^μ from λ_2 to λ_1 along the geodesic. The Jacobi propagator is an 8×8 matrix given by

$$U(\lambda_2, \lambda_1) = \mathcal{P} \exp \left(\int_{\lambda_1}^{\lambda_2} \begin{pmatrix} 0 & 1 \\ P(\lambda_1, \lambda) \mathcal{R}(\lambda) P(\lambda, \lambda_1) & 0 \end{pmatrix} d\lambda \right), \quad (3.7)$$

where $\mathcal{R}(\lambda)^\mu{}_\sigma$ denotes the 4×4 matrix $R^{(0)\mu}{}_{\nu\rho\sigma} k^{(0)\nu} k^{(0)\rho}$ evaluated at $x^{(0)}(\lambda)$, and 0 and 1 denote the 4×4 zero and identity matrices, respectively. The Jacobi propagator serves as a Green's function for the Jacobi equation in the background spacetime. More information about these objects can be found in [20, 22].

The solution for $x^{(a)\mu}(\lambda)$ and $k^{(a)\mu}(\lambda)$ at some affine parameter λ_2 can now be obtained from their values at some fixed affine parameter λ_1 via

$$\begin{aligned} \left(\begin{array}{c} P(\lambda_1, \lambda_2) x^{(a)}(\lambda_2) \\ \frac{d}{d\lambda_2} [P(\lambda_1, \lambda_2) x^{(a)}(\lambda_2)] \end{array} \right) &= U(\lambda_2, \lambda_1) \left(\begin{array}{c} x^{(a)}(\lambda_1) \\ [\frac{d}{d\lambda} [P(\lambda_1, \lambda) x^{(a)}(\lambda)]]_{\lambda=\lambda_1} \end{array} \right) \\ &+ \int_{\lambda_1}^{\lambda_2} U(\lambda_2, \lambda) \begin{pmatrix} 0 \\ P(\lambda_1, \lambda) f^{(a)}(\lambda) \end{pmatrix} d\lambda, \end{aligned} \quad (3.8)$$

the integral being taken over the zeroth order geodesic, $x^{(0)\mu}(\lambda)$. The program for recursive calculation of the $x^{(a)\mu}(\lambda)$ is now established: having obtained $x^{(a-1)\mu}(\lambda)$ we can solve for $f^{(a)\mu}$ and thus obtain $x^{(a)\mu}(\lambda)$ from (3.8). The recursion starts by solving the geodesic equation of the background for some $x^{(0)\mu}(\lambda)$ and then calculating its associated parallel and Jacobi propagators.

The parallel and Jacobi propagators for the radial, null geodesics of $g_{\mu\nu}^{(0)}$, the conformally transformed Robertson-Walker metric, were obtained in [23] and are written

$$P(\lambda_2, \lambda_1)^\mu{}_\nu = \begin{pmatrix} 1 & 0_j \\ 0^i & \frac{\gamma(\lambda_2)}{\gamma(\lambda_1)} \delta_j^i \end{pmatrix} \quad (3.9)$$

and

$$U(\lambda_2, \lambda_1) = \begin{pmatrix} \cos_\kappa(\lambda_2 - \lambda_1) J & \sin_\kappa(\lambda_2 - \lambda_1) J \\ -\kappa \sin_\kappa(\lambda_2 - \lambda_1) J & \cos_\kappa(\lambda_2 - \lambda_1) J \end{pmatrix} + \begin{pmatrix} (1 - J) & (\lambda_2 - \lambda_1)(1 - J) \\ 0 & (1 - J) \end{pmatrix} \quad (3.10)$$

respectively. In (3.10) J is a 4×4 matrix given by

$$J^\mu{}_\sigma = \begin{pmatrix} 0 & 0_j \\ 0^i & \delta_j^i - e^i e_j \end{pmatrix}. \quad (3.11)$$

Given a background geodesic specified by the direction cosines e^i , any three-vector $v^i(\lambda)$ may be decomposed into the sum of a longitudinal part $v_\parallel^i(\lambda)$ pointing along the geodesic and a transverse part $v_\perp^i(\lambda)$ perpendicular to the geodesic (in the spacelike hypersurface of the background), where

$$\begin{aligned} v_\parallel^i(\lambda) &= e^i e_j v^j(\lambda), \\ v_\perp^i(\lambda) &= (\delta_j^i - e^i e_j) v^j(\lambda). \end{aligned} \quad (3.12)$$

Thus, the matrix J serves to project a four-vector into the plane transverse to the photon direction in the comoving spatial hypersurfaces.

The relatively simple form of these propagators for the case of a Robertson-Walker metric allows us to obtain the perturbed geodesic and wavevector from (3.8) immediately. Imposing the boundary conditions (2.9) at the observer, we obtain

$$\begin{aligned} x^{(a)0}(\lambda) &= (\lambda - \lambda_{\mathcal{O}}) k^{(a)0}(\lambda_{\mathcal{O}}) + \int_{\lambda_{\mathcal{O}}}^{\lambda} (\lambda - \lambda') f^{(a)0}(\lambda') d\lambda', \\ x_\parallel^{(a)i}(\lambda) &= \gamma(\lambda) \int_{\lambda_{\mathcal{O}}}^{\lambda} (\lambda - \lambda') \gamma^{-1}(\lambda') f_\parallel^{(a)i}(\lambda') d\lambda', \\ x_\perp^{(a)i}(\lambda) &= \gamma(\lambda) \int_{\lambda_{\mathcal{O}}}^{\lambda} \sin_\kappa(\lambda - \lambda') \gamma^{-1}(\lambda') f_\perp^{(a)i}(\lambda') d\lambda', \\ k^{(a)0}(\lambda) &= k^{(a)0}(\lambda_{\mathcal{O}}) + \int_{\lambda_{\mathcal{O}}}^{\lambda} f^{(a)0}(\lambda') d\lambda', \\ k_\parallel^{(a)i}(\lambda) &= \gamma(\lambda) \int_{\lambda_{\mathcal{O}}}^{\lambda} \left[1 - \frac{\kappa r(\lambda)}{2} (\lambda - \lambda') \right] \gamma^{-1}(\lambda') f_\parallel^{(a)i}(\lambda') d\lambda', \\ k_\perp^{(a)i}(\lambda) &= \gamma(\lambda) \int_{\lambda_{\mathcal{O}}}^{\lambda} \left[\cos_\kappa(\lambda - \lambda') - \frac{\kappa r(\lambda)}{2} \sin_\kappa(\lambda - \lambda') \right] \gamma^{-1}(\lambda') f_\perp^{(a)i}(\lambda') d\lambda'. \end{aligned} \quad (3.13)$$

These expressions are valid for any metric perturbation; the specific perturbation is encoded in the vectors $f^{(a)\mu}$. Note that we have written the integrals as proceeding backwards along the path from the observer to the point λ on the background path.

IV. Scalar Perturbations in the Longitudinal Gauge

In this section we carry out the program described above to second order for the metric perturbation $h_{\mu\nu}$ given by

$$h_{\mu\nu}dx^\mu dx^\nu = -2\phi d\eta^2 - 2\psi\gamma^{-2}(dx^2 + dy^2 + dz^2) \quad (4.1)$$

describing scalar perturbations in the longitudinal gauge [24]. We note that in this gauge, ϕ and ψ coincide with the gauge invariant metric variables of [24], Φ and Ψ . This will allow us to obtain gauge invariant expressions for the observables of interest by replacing ϕ with Φ and ψ with Ψ in our final formulae. Of course, only the first order expressions are rendered gauge invariant because Φ and Ψ are themselves gauge invariant only to first order.

To compute the perturbation vectors $f^{(a)\mu}$ we need to calculate the Christoffel symbols to various orders. These are given by

$$\begin{aligned} \Gamma^{(0)0}_{\sigma\alpha} &= 0 \\ \Gamma^{(0)i}_{0\alpha} &= 0 \\ \Gamma^{(0)i}_{jk} &= -\frac{\kappa}{2\gamma}(\delta_{ik}x_j^{(0)} + \delta_{ij}x_k^{(0)} - \delta_{jk}x_i^{(0)}) \\ \Gamma^{(1)0}_{0\alpha} &= \partial_\alpha\phi \\ \Gamma^{(1)0}_{ij} &= -\frac{\partial_0\psi}{\gamma^2}\delta_{ij} \\ \Gamma^{(1)i}_{00} &= \gamma^2\partial_i\phi \\ \Gamma^{(1)i}_{0j} &= -\partial_0\psi\delta_{ij} \\ \Gamma^{(1)i}_{jk} &= -\delta_{ik}\partial_j\psi - \delta_{ij}\partial_k\psi + \delta_{jk}\partial_i\psi \\ \Gamma^{(2)0}_{0\alpha} &= -2\phi\partial_\alpha\phi \\ \Gamma^{(2)0}_{ij} &= \frac{2\phi\partial_0\psi}{\gamma^2}\delta_{ij} \\ \Gamma^{(2)i}_{00} &= 2\gamma^2\psi\partial_i\phi \\ \Gamma^{(2)i}_{0j} &= -2\psi\partial_0\psi\delta_{ij} \\ \Gamma^{(2)i}_{jk} &= -2\psi(\delta_{ik}\partial_j\psi + \delta_{ij}\partial_k\psi - \delta_{jk}\partial_i\psi) . \end{aligned} \quad (4.2)$$

Calculation of the first order vector $f^{(1)\mu}$ proceeds straightforwardly, using the nor-

malization $g_{ij}^{(0)} k^{(0)i} k^{(0)j} = \delta_{ij} e^i e^j = 1$. We find that

$$\begin{aligned} f^{(1)0} &= \partial_0(\phi + \psi) - 2 \frac{d\phi}{d\lambda} , \\ f_{\parallel}^{(1)i} &= -k^{(0)i} k^{(0)j} \partial_j(\phi + \psi) + 2k^{(0)i} \frac{d\psi}{d\lambda} , \\ f_{\perp}^{(1)i} &= (k^{(0)i} k^{(0)j} - g^{(0)ij}) \partial_j(\phi + \psi) . \end{aligned} \quad (4.3)$$

According to (2.16), the only second-order quantity (as distinguished from products of first-order quantities) which enters the formula for $\widehat{\delta T}^{(2)}$ is the timelike component of the wavevector, $k^{(2)0}$. We therefore do not need to calculate the entire second-order force vector, but only the timelike component. In doing so we make use of the decomposition of the directional derivative of a scalar along the path into partial derivatives,

$$\frac{d\phi}{d\lambda} = \partial_0 \phi + k^{(0)i} \partial_i \phi \quad (4.4)$$

(since $k^{(0)0} = 1$). Another relatively straightforward calculation yields

$$\begin{aligned} f^{(2)0} &= -2 \frac{d}{d\lambda} \left(k^{(1)0} \phi + x^{(1)\sigma} \partial_{\sigma} \phi \right) + 2k^{(1)0} \partial_0(\phi - \psi) \\ &\quad + x^{(1)\sigma} \partial_{\sigma} \partial_0(\phi + \psi) + 2(\phi + \psi) \partial_0 \psi . \end{aligned} \quad (4.5)$$

Substituting (4.3) into (3.13), we obtain the first order perturbed geodesic:

$$\begin{aligned} x^{(1)0}(\lambda) &= (\lambda - \lambda_{\mathcal{O}})(\phi - \psi)_{\mathcal{O}} + \int_{\lambda_{\mathcal{O}}}^{\lambda} [-2\phi + (\lambda - \lambda') \partial_0(\psi + \phi)] d\lambda' , \\ x_{\parallel}^{(1)i}(\lambda) &= (\lambda - \lambda_{\mathcal{O}})(\phi - \psi)_{\mathcal{O}} k^{(0)i}(\lambda) + k^{(0)i}(\lambda) \int_{\lambda_{\mathcal{O}}}^{\lambda} [(\psi - \phi) + (\lambda - \lambda') \partial_0(\psi + \phi)] d\lambda' , \\ x_{\perp}^{(1)i}(\lambda) &= \gamma(\lambda) \int_{\lambda_{\mathcal{O}}}^{\lambda} \sin_{\kappa}(\lambda - \lambda') \gamma(\lambda') [e^i e^j - \delta^{ij}] \partial_j(\phi + \psi) d\lambda' . \end{aligned} \quad (4.6)$$

The explicit construction for the wavevector perturbation, $k^{(1)}(\lambda)$, may be obtained either by differentiation of (4.6) above or from (3.13) directly. In either case

$$\begin{aligned} k^{(1)0}(\lambda) &= (\phi - \psi)_{\mathcal{O}} - 2\phi_{\lambda} - I_{\text{ISW}}(\lambda) , \\ k_{\parallel}^{(1)i}(\lambda) &= -\frac{\kappa r(\lambda)}{2} x_{\parallel}^{(1)i}(\lambda) + k^{(0)i}(\lambda) [(\phi - \psi)_{\mathcal{O}} - (\phi - \psi)_{\lambda} - I_{\text{ISW}}(\lambda)] , \\ k_{\perp}^{(1)i}(\lambda) &= \gamma(\lambda) \int_{\lambda_{\mathcal{O}}}^{\lambda} \left[\cos_{\kappa}(\lambda - \lambda') - \frac{\kappa r(\lambda)}{2} \sin_{\kappa}(\lambda - \lambda') \right] \gamma(\lambda') [e^i e^j - \delta^{ij}] \partial_j(\phi + \psi) d\lambda' , \end{aligned} \quad (4.7)$$

where the integral

$$I_{\text{ISW}}(\lambda) = - \int_{\lambda_{\mathcal{O}}}^{\lambda} \partial_0(\psi + \phi) d\lambda' \quad (4.8)$$

represents the conventional integrated Sachs-Wolfe effect.

As noted above, to compute the second-order effect we need only the time component of the second-order wavevector; this is given by

$$k^{(2)0}(\lambda) = -\frac{1}{2}(\phi + \psi)_{\mathcal{O}}^2 - 2(x^{(1)\mu} \partial_{\mu} \phi + k^{(1)0} \phi)_{\lambda} + I_2(\lambda) , \quad (4.9)$$

where the integral I_2 is defined as

$$I_2(\lambda) = \int_{\lambda_{\mathcal{O}}}^{\lambda} \left[2k^{(1)0} \partial_0(\phi - \psi) + 2(\phi + \psi) \partial_0 \psi + x^{(1)\mu} \partial_{\mu} \partial_0(\phi + \psi) \right] d\lambda' . \quad (4.10)$$

Having obtained the perturbed geodesics and wavevectors in the longitudinal gauge, it remains only to substitute into (2.16) to obtain final expressions for the temperature anisotropy. At first order we recover the conventional Sachs-Wolfe result,

$$\widehat{\delta T}^{(1)} = (\phi + \tau)_{\bar{\mathcal{E}}} - \phi_{\mathcal{O}} + I_{\text{ISW}}(\lambda_{\bar{\mathcal{E}}}) , \quad (4.11)$$

where once again the notation $\bar{\mathcal{E}}$ means that quantities are evaluated at the position and direction of the intersection of the background geodesic with the surface of emission. The second order anisotropy, the main result of this paper, is given by

$$\begin{aligned} \widehat{\delta T}^{(2)} = & \frac{3}{2} \phi_{\mathcal{O}}^2 - \phi_{\mathcal{O}} \phi_{\bar{\mathcal{E}}} - \phi_{\mathcal{O}} \tau_{\bar{\mathcal{E}}} - \frac{1}{2} \phi_{\bar{\mathcal{E}}}^2 + \phi_{\bar{\mathcal{E}}} \tau_{\bar{\mathcal{E}}} \\ & - (2\phi_{\mathcal{O}} - \psi_{\mathcal{O}} - \phi_{\bar{\mathcal{E}}} - \tau_{\bar{\mathcal{E}}} - I_{\text{ISW}}(\lambda_{\bar{\mathcal{E}}})) I_{\text{ISW}}(\lambda_{\bar{\mathcal{E}}}) - I_2(\lambda_{\bar{\mathcal{E}}}) \\ & + \left(x_{\perp}^{(1)i} + I_{\text{TD}} k^{(0)i} \right)_{\bar{\mathcal{E}}} \partial_i(\phi + \tau)_{\bar{\mathcal{E}}} + x^{(1)0} \partial_0(\phi + \psi)_{\bar{\mathcal{E}}} + d^{(1)i} \frac{\partial \tau}{\partial d^i} , \end{aligned} \quad (4.12)$$

where the integral

$$I_{\text{TD}}(\lambda) = \int_{\lambda_{\mathcal{O}}}^{\lambda} (\phi + \psi) d\lambda' \quad (4.13)$$

is the Shapiro time delay along the path.

An accurate appraisal of the magnitudes of the various terms contained in (4.12) would require knowledge of the initial conditions and evolution of the perturbations ϕ and ψ , including nonlinear effects. This information is model-dependent, and we will not

attempt such a task here. It is nevertheless possible to remark on the possible importance of the different effects to observations, based simply on the form in which they appear.

The quantities $\phi_{\mathcal{O}}$, $\psi_{\mathcal{O}}$, $\phi_{\bar{\mathcal{E}}}$, $\tau_{\bar{\mathcal{E}}}$ and $I_{\text{ISW}}(\lambda_{\bar{\mathcal{E}}})$ are all small ($\leq 10^{-5}$) in conventional models of structure formation. Therefore the terms in (4.12) which are written as products of these numbers are even smaller, and should not contribute to the anisotropy at an observable level. Similarly the term $d^{(1)i}(\partial\tau/\partial d^i)$ will typically be the product of two small quantities, and may be neglected. Therefore the potentially interesting terms are those involving the separation vector $x^{(1)\mu}(\lambda_{\bar{\mathcal{E}}})$ (which is not necessarily small) and the integrals $I_{\text{TD}}(\lambda_{\bar{\mathcal{E}}})$ and $I_2(\lambda_{\bar{\mathcal{E}}})$.

The term $x_{\perp}^{(1)i}\partial_i(\phi + \tau)_{\bar{\mathcal{E}}}$ is due to the transverse deflection of the photons by sources between us and the surface of emission; this effect has been studied previously in investigation of the impact of gravitational lenses on CMB anisotropy [25-36]. (The processing of CMB anisotropy by lensing is second order since both the lens angle and the initial fluctuations being processed are themselves first order in our accounting scheme.) While the effect of lensing on the CMB perturbation spectrum has been somewhat controversial, it can play an observable role on small angular scales. The fact that this second-order effect may be significant can be thought of as a consequence of the fact, noted in the Introduction, that the existence of large distance scales in the problem can enhance higher-order effects; in this case the transverse deflection, given approximately by the product of the distance travelled times the lens angle, builds up as the photon travels along its trajectory.

The term $I_{\text{TD}}k^{(0)i}\partial_i(\phi + \tau)_{\bar{\mathcal{E}}}$ is the longitudinal equivalent of the transverse lensing term. It arises from the time delay effect of the lenses, which alters the spacelike distance between the observer and the point where the photon path intersects the surface of emission. The qualitative effect of this term is similar to that of the transverse lensing term, although its magnitude is expected to be smaller; for typical lens systems, the longitudinal deflection is smaller than the transverse deflection by a factor proportional to the lens angle (*i.e.*, by several orders of magnitude).

The term $x^{(1)0}\partial_0(\phi + \psi)_{\bar{\mathcal{E}}}$ arises because the difference in affine parameter between observer and surface of last scattering differs along the true and background paths. It is similar in structure to the effects discussed in the previous two paragraphs, but is presumably smaller since the time derivatives of the potentials are typically smaller than the spatial derivatives.

Finally the integral $I_2(\lambda_{\bar{\epsilon}})$ contains three terms. The first two terms appear small; they are integrals of products of two small quantities, and furthermore contain time derivatives which are typically suppressed with respect to spatial derivatives. The third term represents a correction to the ISW effect, taking into account that the perturbations along the first-order path differ from those along the background path. In cold dark matter models with $\Omega = 1$ and adiabatic density perturbations, the ISW effect itself is smaller than the conventional Sachs-Wolfe term [8, 37-48], and the correction described here is presumably smaller still; nevertheless, it is possible that observations of the CMB will reach a level of precision at which this term should be taken into account. Moreover, time derivatives and the ISW effect can be important in models of structure formation based on topological defects [12-15], open universe models [49-54], and models with an appreciable cosmological constant [55, 56]. In these types of universes the new terms represented by $I_2(\lambda_{\bar{\epsilon}})$ could play a role analogous to that of gravitational lensing in adiabatic CDM models.

V. Conclusions

We have computed the anisotropy induced in the cosmic microwave background, due to gravitational effects, to second order in a given metric perturbation. For an arbitrary perturbation, our results are given by the basic equation (2.16) plus the solutions (3.13) for the perturbed geodesic and wavevector, where the forcing vectors to first and second order are given by (3.5). In the case of scalar perturbations in the longitudinal gauge, these results may be combined into the single compact formula (4.12).

Our results are reassuring for studies to date of CMB anisotropy, in that they do not reveal any new effects which are likely to dominate the anisotropy spectrum on any scale. An informal examination of our final expressions indicates that the effect most likely to be observable is that due to (transverse) gravitational lensing, which has already been the subject of some attention in the literature. As both theoretical and observational studies of the CMB increase in accuracy and sophistication, however, we feel it is important to know the precise form of the effects we have explored.

With the basic framework in hand, there is clearly room for future work along these lines. One direction would be to investigate a wider class of perturbations (*i.e.*, vector and tensor modes), as well as to study carefully the second-order metric perturbation itself. An equally important task is to examine the effects we have described more quantitatively, in

the context of a specific and detailed model of structure formation; only then could we be completely confident in our understanding of the role played by second-order perturbations in CMB anisotropy.

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Figure Caption

Figure One. This figure shows the observer at location $x^\mu(\lambda_{\mathcal{O}})$, the hypersurface of last scattering at $\eta = \eta_{\mathcal{E}}$, and various paths connecting the two. The true geodesic in the perturbed metric is $x^\mu(\lambda)$, while the background geodesic is $x^{(0)\mu}(\lambda)$. Adding the deviation vectors $x^{(1)\mu}(\lambda)$ and $x^{(2)\mu}(\lambda)$ to the background path yields increasingly accurate approximations to the true path. The spacelike deviation vectors $p^{(a)i}$ are to be distinguished from the $x^{(a)\mu}$, since the latter generally do not lie in hypersurfaces of constant conformal time.

